GREATER NUMBER OF LARGER PIECES: A STRATEGY TO PROMOTE PROSPECTIVE TEACHERS' FRACTION NUMBER SENSE DEVELOPMENT

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Prospective teachers (PTs) need opportunities to develop fraction number sense, yet little research has explicated how this development occurs. Our research team collaboratively designed a task targeted at helping PTs develop fraction number sense through an exploration of fraction comparison strategies. This paper focuses on developing one particular strategy, which we call Greater Number of Larger Pieces (GLP). We argue that understanding this strategy has the potential to support PTs' number sense, particularly in regards to the measure interpretation of fractions. Analysis of data from two iterations of this task (implemented by five mathematics teacher educators at five US institutions with 124 PTs) showed an improvement in the task's ability to naturally elicit the GLP strategy from PTs. We share our task, results from each iteration, and discuss modifications that we believe led to increased usage of the GLP strategy.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching, Rational Numbers, Teacher Knowledge

Introduction and Background Information

Researchers have argued that number sense is an important part of the mathematical knowledge needed for teaching (Ball, Thames, & Phelps, 2008; Tsao, 2005), and thus, its development should be an integral component of the mathematical preparation of prospective teachers (PTs). Given the prevalence of fraction topics across the K-8 mathematics curriculum, we focus our work on the development of fraction number sense, which Lamon (2012) defines as "an intuition that helps [students] make appropriate connections, determine size, order, and equivalence, and judge whether answers are or are not reasonable" (p. 136). Lamon argues that this intuition is especially important for teachers to develop, as they will need it to evaluate the appropriateness of student reasoning. Yet research shows that PTs often exhibit particularly weak and procedurally-oriented thinking in terms of fractions (Tobias et al., 2014; Yang, Reys, & Reys, 2009).

In order to work proficiently with fractions, students and teachers should be familiar with a variety of fraction interpretations, rather than focusing solely on the traditionally-taught *part-whole* interpretation (Kieren, 1976; Lamon, 2012; Thompson & Saldanha, 2003). Watanabe (2007) states that "when students' understanding of fractions is limited to the part-whole meaning, it is doubtful that they understand fractions as numbers" (p. 57). Busi and colleagues (2015) note, for example, that considering 3/5 as three parts out of five can lead to a non-sensical interpretation of improper fractions, such as 7/5, as seven parts out of five.

Recommendations in the *Common Core State Standards for Mathematics* (CCSSM) in the United States (National Governors Association [NGA] & Council of Chief State School Officers [CCSSO], 2010), and throughout international curricula (Son, Lo, & Watanabe, 2015; Watanabe, 2006; 2007), suggest that viewing a fraction as a measure can help overcome some of the limitations of the *part-whole* interpretation. This view is exemplified by the following third-grade CCSSM content standard (3.NF.A.1): *Understand a fraction 1/*b *as the quantity formed by 1 part when a*

whole is partitioned into b *equal parts; understand a fraction* a/b *as the quantity formed by* a *parts of size 1/*b. The *fraction-as-a-measure* interpretation can support children's development of fraction addition and subtraction knowledge (Son, Lo, Watanabe, 2015) by helping students recognize how adding fractions with like denominators is adding additional iterations of the unit fractions that comprise the original fractions. For example, if one sees 3/7 + 2/7 as 3 pieces of 1/7 and 2 pieces of 1/7, then the combined result of 5 pieces of 1/7, or 5/7, makes sense. Such understanding could help students avoid a common error of adding across the numerators and denominators, i.e., 3/7 + 2/7 = 5/14 (Mack, 1995). These ideas can then be extended to the addition and subtraction of fractions with unlike denominators (McNamara, 2015).

A Task to Develop Fraction Number Sense

This paper reports on our efforts to help PTs develop fraction number sense, including the ability to interpret fractions as measures, through the study of comparing and ordering fractions. A group of six mathematics teacher educators developed and enacted a fraction comparison task, based on a task designed for fifth-graders, in their mathematics content courses for PTs (Tobias et al., 2014). Our goal was to help PTs shift their perspectives on fractions from a *part-whole* to a *measure* interpretation, and in doing so, begin to see fractions as numbers. Research and policy recommendations highlight the importance of providing learners with repeated opportunities to grapple with problems and generate their own solution strategies instead of applying a strategy made explicit by an instructor (Conference Board of Mathematical Sciences [CBMS], 2012; Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996). To this end, we created a task consisting of ten fraction comparison problems designed to help PTs develop understandings of several fraction comparison strategies beyond *common denominators*, the strategy with which they are traditionally most familiar (Olanoff, Lo, & Tobias, 2014).

Although PTs were given the freedom to construct their own fraction comparison strategies, the task was designed to support the development of several predetermined fraction comparison strategies identified in the literature, specifically: *Same Size Pieces* (SSP; also known as common denominators), *Same Number of Pieces* (SNP; also known as common numerators), and *Comparing to a Benchmark Value* (BV) (Lamon, 2012; NGA & CCSSO, 2010). Given our goal of helping teachers develop understandings of fractions as measures, we sought to introduce an additional strategy that we refer to as *Greater Number of Larger Pieces* (GLP). In this strategy, one uses the measure interpretation of fractions to consider each fraction as a certain number of equal-sized pieces. For example, when comparing 18/25 to 16/27, one can interpret 18/25 as eighteen fractional pieces each of size 1/25 and 16/27 as sixteen pieces each of size 1/27. Since pieces of size 1/25 are larger than those of size 1/27 and there is a greater number of them (18 > 16), one can conclude that 18/25 > 16/27. The use of the GLP strategy requires the simultaneous coordination of two quantities - one referring to the *number* of fractional pieces and one referring to the *size* of those pieces.

Below, we describe the implementation of two iterations of our fraction comparison task in mathematics content courses for PTs. We present data analysis focused on PTs' abilities to successfully implement the GLP strategy to solve individual fraction comparison problems and the task's ability to elicit the strategy from PTs. We will discuss GLP-related modifications that were made to the first version of the task based on our data analysis and the effects of those changes. For a more detailed explanation of the task design process, data analysis, and findings pertaining to the task as a whole, see Thanheiser et al. (2016).

Task Version 1

The first version of our task included ten fraction comparison problems designed to elicit our four targeted fraction comparison strategies noted above: SSP, SNP, BV, and GLP. Some problems also required finding equivalent fractions (EF) in conjunction with one of the four strategies. PTs

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were instructed to circle the larger fraction and to provide a sense-making justification for their choice. Table 1 presents each fraction comparison problem, as well as the strategy we intended for each problem to elicit.

Table 1. Task Version 1: Ten fraction comparison problems with intended strategies (bracketed numerals following BV indicate intended benchmark values). Underlining indicates the greater fraction.

Problem	Fractions to compare	Intended strategy	Problem	Fractions to compare	Intended strategy
#1	1/2 vs. <u>17/31</u>	BV [½], EF-SSP, or EF-SNP	#6	13/15 vs. <u>17/19</u>	BV [1]
#2	<u>2/17</u> vs. 2/19	SNP	#7	5/6 vs. <u>6/5</u>	BV [1]
#3	4/7 vs. <u>9/14</u>	EF-SSP	#8	7/10 vs. <u>8/9</u>	GLP
#4	3/7 vs. <u>6/11</u>	BV [½] or EF-SNP	#9	1/4 vs. <u>25/99</u>	BV [¼] or EF-SNP
#5	8/9 vs. <u>12/13</u>	BV [1]	#10	<u>24/7</u> vs. 34/15	BV [3]

The data analysis included in this paper will focus on problem #8 (7/10 vs. 8/9) since this was the only problem intended to elicit the GLP strategy.

Task Implementation and Data Collection

We launched the task above by asking PTs to list everything they knew about 7/8. We also gave them the following two prompts to work on in small groups:

1. Keeping the denominator the same, find 3 fractions that are greater than 7/8, and 3 fractions that are less than 7/8.
2. Keeping the numerator the same, find 3 fractions that are greater than 7/8, and 3 fractions that are less than 7/8.

Following small group work, each instructor facilitated a brief whole-class discussion in which PTs articulated and justified their thinking.

One of the primary goals of the launch activity was to help PTs begin the transition towards interpreting fractions as measures. For example, many PTs' responses focused on the idea of 7/8 meaning 7 out of 8 parts, which gave instructors the opportunity to highlight additional interpretations, such as 7 pieces of size ⅛. Furthermore, some instructors asked PTs to justify their answers to the prompts above to solidify the relationship between fractions that have the same denominators and those with same numerators.

Following the launch, PTs were given handouts containing the ten comparison problems and instructed to work on the problems either individually or in small groups, without the use of calculators. PTs were encouraged to think beyond using the SSP strategy and apply their understanding of fractions as numbers to find alternative ways of determining the larger fraction in each pair. Instructors allowed the PTs to work on the task for 30-60 minutes before bringing them all together for a class discussion on their solutions and strategies. PTs' written work on the task was

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collected and copied before the whole group discussion where individual strategies were explicated. This student work on the task prior to the whole class discussion comprises the data for this paper.

Data Analysis and Results - Round 1

Version 1 of our task was implemented in the spring of 2013 by three authors with 61 PTs in four mathematics content courses for elementary PTs across three universities. PTs' written work was analyzed for correctness of solutions, strategies applied, and quality of explanations. While this task proved successful at helping PTs elicit many of the intended fraction comparison strategies, it was not particularly successful at eliciting GLP. In fact, while 51 of the 52 (98%) PTs who answered problem #8 (7/10 vs. 8/9) were able to correctly identify the greater fraction, only three responses (6%) used the intended (GLP) strategy. An additional 10 responses (19%) included the use of valid strategies, such as finding common denominators or converting the fractions to decimals or percents. However, we found that the remaining 75% of responses offered incorrect or incomplete reasoning, e.g., claiming that 8/9 > 7/10 because 8/9 is "close to 1," while 7/10 is "3 pieces away." While this line of thinking does leverage some intuition about a fraction's magnitude, it does not provide a reasoned explanation for how one knows that 8/9 is closer to 1 than 7/10. Moreover, it does not attend to the *fraction-as-a-measure* interpretation, which is a necessary component of the GLP strategy. Additionally, data revealed that the three PTs who developed GLP were in only two of the four classes; thus the eventual presentation of the strategy in the other two classes had to come from the instructors, as opposed to the knowledge being constructed and shared by the learners.

These results led to us hypothesizing two reasons for problem #8's inability to aid in the natural emergence of the GLP strategy. First, given the rate at which algorithmic procedures were successfully applied, we believe our choice of fractions (7/10 and 8/9) did not compel PTs to reason about the number and size of pieces. It appears that PTs were familiar enough with the chosen fractions to use their intuition about their magnitude to determine the larger fraction. Many PTs applied their understanding of 7/10 = 0.7 or 70% and 8/9's "closeness to 1" to justify that 8/9 > 7/10. It seemed that the PTs did not see a need for additional reasoning or a more robust justification. Second, we believe that the GLP strategy may be more difficult to apply than other fraction comparison strategies (such as SNP) because it requires the simultaneous interpretation and coordination of both the numerators and denominators.

Task Modifications

Since the first version of our task was not as successful in eliciting the GLP strategy as we had hoped, we added three new comparison problems to the task with the goal of providing more opportunities for PTs to think about and develop GLP on their own. First, we added 2/9 vs ⅜ (problem #14), which can be solved using a variety of strategies including the GLP strategy. Eliciting multiple solution strategies is known to be an important characteristic of effective tasks (Stein, Grover, & Henningsen, 1996). Second, we added 2/7 vs ⅜ (problem #11), which is purposefully similar to 2/9 vs ⅜, but cannot be solved using GLP. We added this problem to provide an opportunity for PTs to determine when GLP is and is not applicable, as knowing when a particular strategy is and is not appropriate to use can further deepen PTs' understanding of the strategy (Borich, 2011). Third, we added 18/25 vs 16/27 (problem #15), specifically choosing fractions with larger numerators and larger, relatively prime, denominators to deter PTs from using computation-heavy strategies (e.g., common denominators or common numerators), and instead look for more efficient strategies. A total of five new items were added to the task, resulting in a second iteration containing 15 problems. Problem #8, the original GLP problem (7/10 vs. 8/9), was left unmodified.

We also revised our implementation of the launch activity to better elicit the *fraction-as-a-measure* interpretation. Instructors spent more time discussing PTs' explanations of 7/8 and pressing PTs for complete "sense-making" explanations for why certain fractions were greater than or less

than 7/8. As one instructor noted in her instructor memo, "I really took my time on this and pressed and probed a lot more than I did last spring" (Hillen, instructor memo). For example, explanations such as "7/9 is farther away from 1 than 7/8" were no longer deemed satisfactory; instead, instructors pushed PTs to explain how they knew this was true. When pressed to explain why 7/8 > 7/9, PTs began to recognize that both fractions have the same number of fractional pieces (7), but since eighths are larger than ninths, 7/8 must be greater than 7/9. Similarly, when explaining why 9/8 > 7/8, PTs recognized that 9/8 has a greater number of eighths. In this way, the launch provided opportunities for PTs to begin thinking about fractions as measures.

Data Analysis and Results - Round 2

The second iteration of the task was implemented during the fall of 2013 by four authors (two of whom were involved in the first implementation) in classrooms at four US institutions with a total of 63 PTs. Compared to the 3 GLP-related responses we received on the first version of the task (6% of PTs who answered #8), nearly 15% of the 61 PTs who answered problem #8 on the second version correctly applied the GLP strategy. PTs' success rates were even greater for the two newly-added GLP comparison problems #14 (2/9 vs 3/8) and #15 (18/25 vs 16/27). Of the 52 PTs who answered #14, over 21% used GLP reasoning; and, while only 46 PTs answered #15, the last item on the task, over 41% of them applied GLP.

Table 2. A comparison of success rates for PTs' application of the GLP strategy on the first and second iterations of the task. (N=61 for Iteration 1; N=63 for Iteration 2)

Iteration	Problem #	GLP Problem	# of PTs who answered the question	% of responses received with correct answers	% of responses received using GLP
1	#8	7/10 vs 8/9	55	98.0%	6.0%
2	#8	7/10 vs 8/9	61	96.7%	14.8%
2	#14	2/9 vs 3/8	52	96.2%	21.2%
2	#15	18/25 vs 16/27	46	95.7%	41.3%

It should be noted that Problems #14 and #15 were the last two comparison problems on the 15-item revised task, so low completion rates may be related to PTs not having enough time to attempt them. However, it might also be that PTs intentionally left these items blank because they were unsure about how to approach them. Regardless, we believe that the major increase in the application of GLP on #15 was at least partly due to the problem being specifically designed to discourage other comparison strategies. With the comparatively large numerator and denominator values, and the selection of relatively prime denominators, the use of the common denominator and common numerator strategies becomes quite tedious, especially without the use of the calculator. As such, it appears that PTs were more compelled to seek alternative strategies for solving #15 than #8 or #14, and ended up applying the *fraction-as-a-measure* interpretation by considering each fraction as a number of equal-sized pieces.

Discussion and Conclusions

The data provide evidence that the second iteration of the task was more successful in eliciting the GLP strategy. Out of the 63 PTs who worked on the second iteration of our task, 21 of them (33%) used GLP on at least one problem, 19% used it on two, and 8% used it on all three of the

applicable problems. These findings are in contrast to the first task iteration, during which only 5% of PTs who worked on the task attempted to use GLP. Additionally, at least two PTs in each of the four classes discovered the GLP strategy on their own, which provided an opportunity for PTs to lead discussions about the strategy with their classmates.

One task modification strategy that has potential for supporting PT learning is creating problems that lend themselves to particular solution strategies while discouraging the use of alternate strategies. Problem #15 (18/25 vs 16/27) is an example of such a problem; PTs were most successful in correctly using the GLP strategy here. Although the benefits of constructing mathematical tasks that elicit multiple strategies are well known (Stein, Grover, & Henningsen, 1996), our work suggests a nuanced approach to task design. In order to elicit particular ideas or strategies, there may be a benefit to constructing some problems that narrow the field of possible solution strategies to those under investigation. In this way, PTs are forced to abandon certain strategies that may not support reasoning and sense making.

Nevertheless, even for problem #15 the majority (58.7%) of the 46 PTs who answered the question opted to use alternative fraction comparison strategies including applying common denominators or common numerators, and comparing the distance of the fractions from ½ or 1. Five PTs (11%) correctly identified the larger fraction, but gave no explanation for their answers. It is interesting to note that seven PTs (15%) showed evidence of some GLP-related thinking, but either their thinking was incorrect or their responses were too incomplete to conclude that the GLP strategy had been successfully applied. For example, some PTs recognized that pieces of size 1/25 are larger than pieces of size 1/27, but they did not attend to the number of fractional pieces in either fraction. Others recognized that 18/25 has a greater number of fractional pieces than 16/27, but did not address the size of those pieces. This result supports the contention that the GLP strategy is challenging for PTs because it requires simultaneous coordination of both the number (numerator) and size (denominator) of the fractional pieces in each fraction being compared.

On a positive note, follow-up analysis suggests that this task can serve as a useful introduction to the GLP strategy. Data from final exams in three of the four classes showed that 78.6% (44 out of 56) of the PTs were able to correctly use a measure interpretation of fractions to justify why the GLP strategy cannot be used to compare fractions such as 27/29 and 31/33.

The evolution of PTs' fraction number sense is critical to their development of mathematical knowledge needed for teaching (Lamon, 2012). While this idea is clearly detailed in the literature, there is little research pertaining to *how* teacher educators can help PTs develop this knowledge. The results of our study provide support for the use of the GLP strategy as one way in which teacher educators can begin to facilitate PTs' fraction number sense. We also provide task design suggestions for eliciting the GLP strategy. Though we found more PTs successful on the second iteration of the task, we recognize that the GLP strategy is complex and may not be learned easily. Thus, future research will be needed to determine additional ways to increase the number of PTs that develop the GLP strategy as well as designing, implementing, and analyzing tasks that will encourage PTs to apply their fraction number sense to develop conceptual and sense-making methods for operating with fractions.

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